Journal of Engineering Physics and Thermophysics, Vol. 76, No. 6, 2003

MATHEMATICAL MODEL OF A RECUPERATIVE HEAT EXCHANGER IN A TWO-DIMENSIONAL FORMULATION

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UDC 621.56:519.8

A two-dimensional model of a recuperative heat exchanger which allows calculation of flows through a heat-transfer wall and temperature at any point of it has been developed. The results of the calculations are in satisfactory agreement with experimental data.

In studying thermal processes, in [1–4] a model of convective heat transfer in the simplest formulation was used. Recently, models which allow for the thermal conductivities of both liquids [5] and walls of the device [6–8] in a longitudinal direction have been developed. In [6], the case of longitudinal thermal conductivity of the partition was considered, whereas in [7, 8], that of all walls was investigated. In the transverse direction, heat transfer in them was still assumed to occur due to convection.

In what follows, we take into account heat transfer in all directions. We assume that the device consists of two channels of constant width which are separated by a thin partition. From the outer sides the channels are bounded by partitions of a specified constant thickness (Fig. 1). Different liquids flow in the two plane channels. One liquid has a high temperature at the inlet and serves as a heat carrier; the other (with a lower temperature) is a coolant. Such an apparatus is used either to heat the coolant or to cool the heat carrier, although these processes occur simultaneously independent of the specific engineering aim.

In our case, the processes of heat conduction take place in solids (channel walls) and in two liquids. The equations of heat conduction under steady-state conditions for solid walls and liquid layers have the form [9]

$$\lambda_i \left(\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} \right) + f_i = 0 \,, \quad i = 1, 3, 5 \,, \tag{1}$$

$$v_j \frac{\partial T_j}{\partial x} = a_j^2 \left(\frac{\partial^2 T_j}{\partial x^2} + \frac{\partial^2 T_j}{\partial y^2} \right) + \frac{f_j}{c_j \rho_j} \,, \quad j = 2, 4 \,. \tag{2}$$

Here $i = 1$ corresponds to the lower wall, $i = 3$ to the central wall (partition), and $i = 5$ to the upper wall; $j = 2$ corresponds to the lower channel filled with liquid and $j = 4$ to the upper channel.

If the liquid in the jth channel flows to the right, then $v_j > 0$; if it flows to the left, then $v_j < 0$.

Differential equations (1) and (2) must be supplemented with boundary conditions which could provide uniqueness of their solution. To solve the problem we use an integral method of straight lines [10] which allows us to impose different boundary conditions on the lower boundary of the wall 1 and on the upper boundary of the wall 5 (Fig. 1), which will virtually not complicate the procedure of obtaining a solution of the whole system.

For the sake of definiteness, we assume that the lower boundary of the wall 1 is thermally insulated and on the upper boundary of the wall 5 convective heat exchange with the surrounding air, whose temperature is T_0, takes place [11, 12]. Hence we obtain two boundary conditions:

Voronezh State Technological Academy, Voronezh, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 76, No. 6, pp. 161–167, November–December, 2003. Original article submitted October 29, 2002; revision submitted March 18, 2003.

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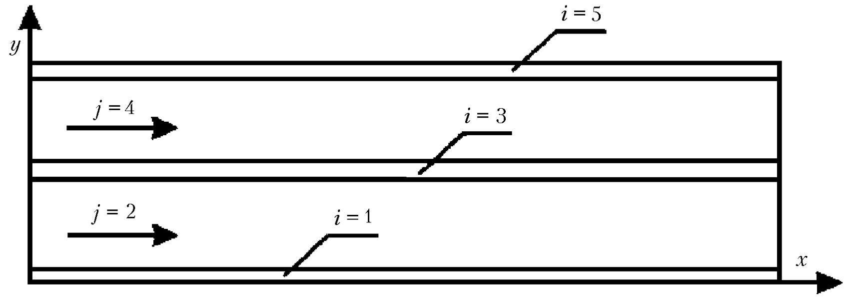

Fig. 1. Schematic of a heat exchanger.

$$\left.\frac{\partial T_1}{\partial y}\right|_{y=0} = 0 \; ; \quad \lambda_5 \left.\frac{\partial T_5}{\partial y}\right|_{y=H_5} + \alpha_5 \left(T_5 \left|_{y=H_5} - T_0\right.\right) = 0 \; . \tag{3}$$

In addition to (3) we must write boundary conditions at the places of contact between the liquids and walls, which follow from the conditions of continuity of temperatures and flows:

$$T_k \left|_{y=H_k}\right. = T_{k+1} \left|_{y=H_k}\right. , \quad \lambda_k \left.\frac{\partial T_1}{\partial y}\right|_{y=H_k} = \lambda_{k+1} \left.\frac{\partial T_{k+1}}{\partial y}\right|_{y=H_k} , \quad H_{k+1} = H_k + h_{k+1} \, , \quad k = 1, ..., 4 \; . \tag{4}$$

Conditions (3) and (4) describe thermal processes in the transverse direction. Boundary conditions on the right and left boundaries of the heat exchanger at $x = 0$ and $x = L$ must be added to them.

In the steady-state regime, the cold liquid at the inlet to the heat exchanger will have the following temperature distribution over the cross section $x = 0$: at points which are closest to the hot liquid, i.e., $y = H_3 + \Delta$, the temperature of the cold liquid will be the highest due to heating by the hot liquid, and at the most distant points, where $y = H_4 - \Delta$, the temperature of the cold liquid can be close to its temperature in the supply branch pipe, i.e., T_4^*. Here and in what follows, Δ denotes a certain small quantity. Due to heat exchange with the cold liquid in the cross section $x = 0$, the hot liquid will have the lowest temperature at $y = H_2 - \Delta$ and the highest temperature at $y = H_1 + \Delta$. The latter can be taken to be the temperature of the hot liquid in the supply branch pipe, i.e., T_2^*. These considerations lead to the conclusion that the boundary conditions in the liquid layers take on the form

$$T_2 \left|_{x=0, y=H_1}\right. = T_2^* \, , \quad T_4 \left|_{x=0, y=H_4}\right. = T_4^* \; . \tag{5}$$

Due to thermal insulation we write the boundary conditions in the liquid layers in the cross section $x = L$ as

$$\left.\frac{\partial T_2}{\partial x}\right|_{x=L, y=H_1} = 0 \, , \quad \left.\frac{\partial T_4}{\partial x}\right|_{x=L, y=H_4} = 0 \; . \tag{6}$$

Below, to solve the problem we use an approximate integral method of straight lines. Therefore, on the end boundaries at $x = 0$ and $x = L$ in solids, using the integrals we can write the averaged boundary conditions as

$$\begin{gathered} \int_{H_{i-1}}^{H_i} \lambda_i \left.\frac{\partial T_i}{\partial x}\right|_{x=0} dy = \int_{H_{i-1}}^{H_i} \alpha_i \left(T_i - T_0\right) \left|_{x=0}\right. dy \, , \\ - \int_{H_{i-1}}^{H_i} \lambda_i \left.\frac{\partial T_i}{\partial x}\right|_{x=L} dy = \int_{H_{i-1}}^{H_i} \alpha_i \left(T_i - T_0\right) \left|_{x=L}\right. dy \; ; \quad i = 1, 3, 5 \; . \end{gathered} \tag{7}$$

Conditions (7) are a consequence of the assumption on convective heat exchange with the surrounding air on the ends of the solid walls 1, 3, and 5.

We assume that the thicknesses of the walls and liquid layers are small. Then, the temperatures T_i in these layers can be represented by the Taylor series, where we restrict ourselves to squares over the variable $(y - y_i^*)$:

$$T_i = m_i(x) + b_i(x)(y - y_i^*) + g_i(x)(y - y_i^*)^2 , \tag{8}$$

where m_i, b_i, and g_i depend only on the coordinate x.

Equations of heat conduction (1) and (2) are solved in the mean over the transverse coordinate y. To do this, we substitute T_i from (8) into (1) and (2) and integrate each equation over the thickness of the corresponding layer. In this case, Eqs. (1) are transformed to the form

$$\int_{H_{i-1}}^{H_i} \left[\lambda_i \left(\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2}\right) + f_i\right] dy = 0 , \quad \lambda_i \left[m_i''(x) + \frac{1}{12} g_i''(x) h_i^2\right] + 2\lambda_i g_i(x) + f_i = 0 , \quad i = 1, 3, 5 . \tag{9}$$

We transform Eqs. (2) in a similar way:

$$\int_{H_{j-1}}^{H_j} v_j \frac{\partial T_j}{\partial x} dy = \int_{H_{j-1}}^{H_j} \left[a_j^2 \left(\frac{\partial^2 T_j}{\partial x^2} + \frac{\partial^2 T_j}{\partial y^2}\right) + \frac{f_j}{c_j \rho_j}\right] dy ,$$

$$v_j \left[m_j'(x) + \frac{1}{12} g_j'(x) h_j^2\right] = a_j^2 \left[m_j''(x) + \frac{1}{12} g_j''(x) h_j^2\right] + 2a_j^2 g_j(x) + \frac{f_j}{c_j \rho_j} , \quad j = 2, 4 . \tag{10}$$

The system of differential equations (9) and (10) consists of five equations. Each of them is linear and of second order relative to ten unknowns m_i, g_i and m_j, g_j. This system must be supplemented by ten equations, which are obtained from (3) and (4):

$$b_1(x) - g_1(x) h_1 = 0 , \quad b_5(x) + g_5(x) h_5 + \frac{\alpha_5}{\lambda_5}\left[m_5(x) + b_5(x)\frac{h_5}{2} + g_5(x)\frac{h_5^2}{4} - T_0\right] = 0 ;$$

$$m_i(x) + b_i(x)\frac{h_i}{2} + g_i(x)\frac{h_i^2}{4} = m_{i+1}(x) - b_{i+1}(x)\frac{h_{i+1}}{2} + g_{i+1}(x)\frac{h_{i+1}^2}{4} , \quad i = 1, ..., 4 ; \tag{11}$$

$$\lambda_i [b_i(x) + g_i(x) h_i] = \lambda_{i+1} [b_{i+1}(x) - g_{i+1}(x) h_{i+1}] , \quad i = 1, ..., 4 .$$

Equations (11) are linear and algebraic. It only remains for us to use expansions (8) for the end boundary conditions (5)–(7):

$$m_j(0) \pm b_j(0)\frac{h_j}{2} + g_j(0)\frac{h_j^2}{4} = T_j^* , \quad m_j'(L) \pm b_j'(L)\frac{h_j}{2} + g_j'(L)\frac{h_j^2}{2} = 0 , \quad j = 2, 4 ;$$

$$\lambda_i \left[m_i'(0) + \frac{1}{12} g_i'(0) h_i^2\right] = \alpha_i \left[m_i(0) + \frac{1}{12} g_i(0) h_i^2 - T_0\right], \quad i = 1, 3, 5 ; \tag{12}$$

$$-\lambda_i \left[m_i'(L) + \frac{1}{12} g_i'(L) h_i^2\right] = \alpha_i \left[m_i(L) + \frac{1}{12} g_i(L) h_i^2 - T_0\right], \quad i = 1, 3, 5 .$$

In Eqs. (12), the plus sign corresponds to $j = 4$ and the minus sign to $j = 2$.

Differential equations (9) and (10) are inhomogeneous equations with the inhomogeneities being constant quantities. To reduce these equations to a homogeneous form, we make the substitution

$$m_i(x) = M_i(x) + m_{i0}\,,\quad b_i(x) = B_i(x) + b_{i0}\,,\quad g_i(x) = A_i(x) + g_{i0}\,,\quad i = 1, \ldots, 5\,, \tag{13}$$

where m_{i0}, b_{i0}, and g_{i0}, which are constant quantities, are partial solutions of system (9)–(11).

After substitution of (13) into (9)–(11) we have

$$m_{i0} = m_{(i+1)0} - b_{(i+1)0}\frac{h_{i+1}}{2} + g_{(i+1)0}\frac{h_{i+1}^2}{4} - b_{i0}\frac{h_i}{2} - g_{i0}\frac{h_i^2}{4}\,,\quad i = 1, \ldots, 4\,;$$

$$m_{50} = T_0 - (b_{50} + g_{50}h_5)\left(\frac{\lambda_5}{\alpha_5} + \frac{h_5}{2}\right),\quad g_{i0} = -\frac{f_i}{2\lambda_i}\,,\quad i = 1, \ldots, 5\,; \tag{14}$$

$$b_{10} = g_{10}h_1\,;\quad b_{(i+1)0} = \frac{\lambda_i}{\lambda_{i+1}}(b_{i0} + g_{i0}h_i) + g_{(i+1)0}h_{i+1}\,,\quad i = 1, \ldots, 4\,.$$

Using (13), from (9)–(11) we have for $A_i(x)$, $B_i(x)$, and $M_i(x)$

$$\lambda_i\left[M_i''(x) + \frac{1}{12}A_i''(x)\,h_i^2\right] + 2\lambda_i A_i(x) = 0\,,\quad i = 1, 3, 5\,;$$

$$v_j\left[M_j'(x) + \frac{1}{12}A_j'(x)\,h_j^2\right] = a_j^2\left[M_j''(x) + \frac{1}{12}A_j''(x)\,h_j^2\right] + 2a_j^2A_j(x)\,,\quad j = 2, 4\,;$$

$$B_1(x) - A_1(x)\,h_1 = 0\,,\quad \lambda_i\,[B_i(x) + A_i(x)\,h_i] = \lambda_{i+1}\,[B_{i+1}(x) - A_{i+1}(x)\,h_{i+1}]\,,\quad i = 1, \ldots, 4\,;$$

$$M_i(x) + B_i(x)\frac{h_i}{2} + A_i(x)\frac{h_i^2}{4} = M_{i+1}(x) - B_{i+1}(x)\frac{h_{i+1}}{2} + A_{i+1}(x)\frac{h_{i+1}^2}{4}\,,\quad i = 1, \ldots, 4\,; \tag{15}$$

$$B_5(x) + A_5(x)\,h_5 + \frac{\alpha_5}{\lambda_5}\left[M_5(x) + B_5(x)\frac{h_5}{2} + A_5(x)\frac{h_5^2}{4}\right] = 0\,.$$

After the substitution

$$M_i(x) = M_{i0}\exp\mu x\,,\quad A_i(x) = A_{i0}\exp\mu x\,,\quad B_i(x) = B_{i0}\exp\mu x\,,\quad i = 1, \ldots, 5\,, \tag{16}$$

Equations (15) are as follows:

$$M_{i0}\mu^2 + \frac{1}{12}A_{i0}\mu^2h_i^2 + 2A_{i0} = 0\,,\quad i = 1, 3, 5\,;$$

$$v_j\left[M_{j0}\mu + \frac{1}{12}A_{j0}\mu h_j^2\right] = a_j^2\left[M_{j0}\mu^2 + \frac{1}{12}A_{j0}\mu^2h_j^2\right] + 2a_j^2A_{j0}\,,\quad j = 2, 4\,;$$

$$B_{10} - A_{10}h_1 = 0\,,\quad \lambda_i\left[B_{i0} + A_{i0}h_i\right] = \lambda_{i+1}\left[B_{(i+1)0} - A_{(i+1)0}h_{i+1}\right],\quad i = 1, \ldots, 4\,; \tag{17}$$

$$M_{i0} + B_{i0}\frac{h_i}{2} + A_{i0}\frac{h_i^2}{4} = M_{(i+1)0} - B_{(i+1)0}\frac{h_{i+1}}{2} + A_{(i+1)0}\frac{h_{i+1}^2}{4}\,,\quad i = 1, \ldots, 4\,;$$

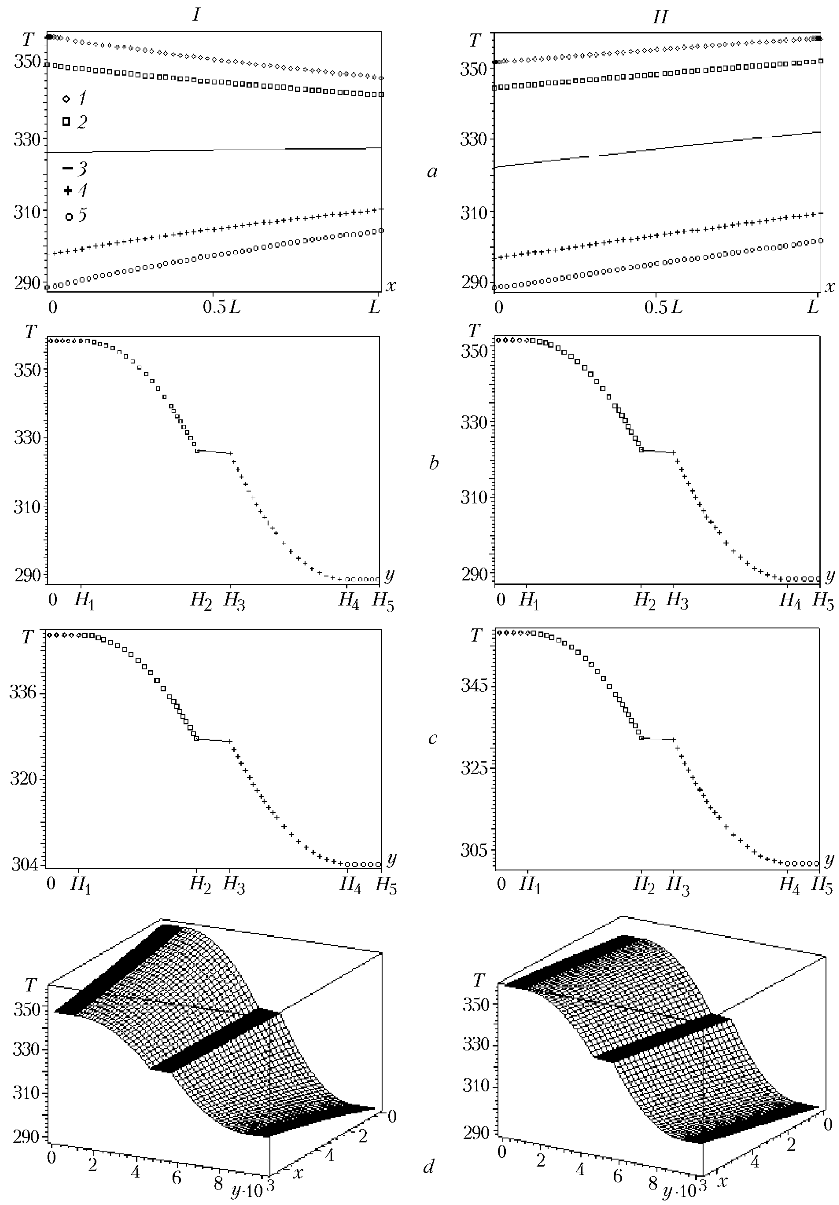

Fig. 2. Temperature distributions in forward (I) and backward (II) flows [a) longitudinal at $y = y_i^*$; b) transverse in the cross section $x = 0$; c, transverse in the cross section $x = L$; d) in the plane xy]: 1) T_1; 2) T_2; 3) T_3; 4) T_4; 5) T_5. T, K; x, y, m.

$$B_{50} + A_{50} h_5 + \frac{\alpha_5}{\lambda_5}\left[M_{50} + B_{50}\frac{h_5}{2} + A_{50}\frac{h_5^2}{4}\right] = 0 .$$

From system (17) we obtain the characteristic equation

$$E_{10}\mu^{10} + E_9\mu^9 + E_8\mu^8 + E_7\mu^7 + E_6\mu^6 + E_5\mu^5 + E_4\mu^4 + E_3\mu^3 + E_2\mu^2 + E_1\mu + E_0 = 0 . \tag{18}$$

The roots μ_n (n = 1, ..., 10) of Eq. (18) are calculated by the Maple 6 program [13]. We find the corresponding values of A_{i0}, M_{i0}, and B_{i0} for each root μ_n. Then the functions $A_i(x)$, $M_i(x)$, and $B_i(x)$ (i = 1, ..., 5) are found from the following equations:

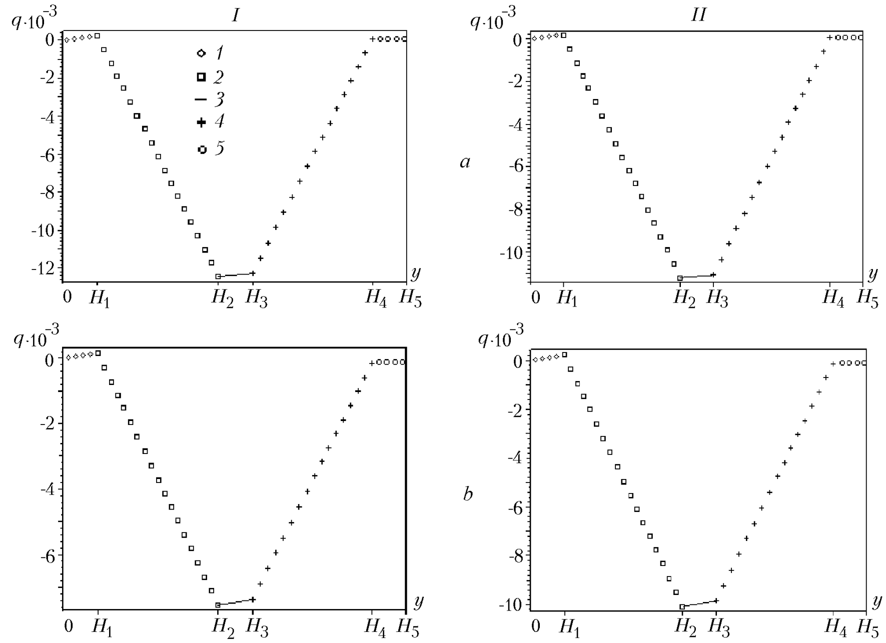

Fig. 3. Distributions of heat-flux density in forward (I) and backward (II) flows (a, in the cross section $x = 0$; b, in the cross section $x = L$): 1) q_1; 2) q_2; 3) q_3; 4) q_4; 5) q_5. q, W/m^2; y, m.

$$A_i(x) = \sum_{n=1}^{10} C_n A_{i0}^{(n)} \exp \mu_n x\,, \quad M_i(x) = \sum_{n=1}^{10} C_n M_{i0}^{(n)} \exp \mu_n x\,,$$

$$B_i(x) = \sum_{n=1}^{10} C_n B_{i0}^{(n)} \exp \mu_n x\,, \quad i = 1, ..., 5\,. \tag{19}$$

To find the constants C_1–C_{10} we use (12) and determine $g_i(x)$, $m_i(x)$, and $b_i(x)$ ($i = 1, ..., 5$) from system (13).

In the case of the backward flow, conditions (5) and (6) are

$$\left.\frac{\partial T_2}{\partial x}\right|_{x=0,y=H_1} = 0\,, \quad T_4\big|_{x=0,y=H_4} = T_4^*\,, \quad T_2\big|_{x=L,y=H_1} = T_2^*\,, \quad \left.\frac{\partial T_4}{\partial x}\right|_{x=L,y=H_4} = 0\,. \tag{20}$$

Solutions of the problem with forward and backward flows of heat carriers are similar. The heat-flux density in the transverse direction for each layer can be calculated by the formula

$$q_i = \lambda_i \frac{\partial T_i}{\partial y} = \lambda_i\,[b_i(x) + 2g_i(x)\,(y - y_i^*)]\,, \quad i = 1, ..., 5\,. \tag{21}$$

The dependences of temperature T and heat-flux density q on the coordinates x and y for different regimes of heat-carrier flows are given in Figs. 2–4.

It is seen from the graphs (Fig. 2) that heat transfer occurs in the following way: at the inlet in the cross section $x = 0$ both liquids immediately participate in mutual heat exchange through the partition and their temperatures T_2 and T_4 have a great difference over the cross section, which lies within the range of wall temperatures [T_1, T_3] and [T_3, T_5], respectively. We assume this information to be of importance. The equalities $T_2 = T_2^*$ and $T_4 = T_4^*$ hold in forward flow in the cross section $x = 0$ and backward flow in the cross sections $x = 0$ and $x = L$ only at points which

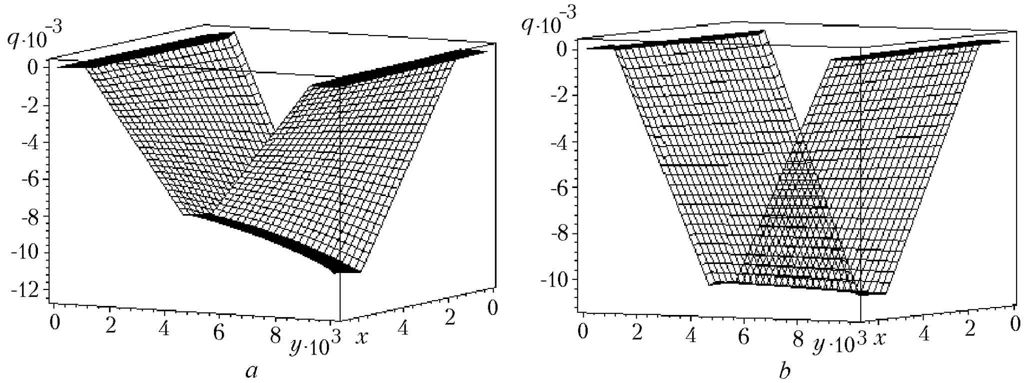

Fig. 4. Distributions of heat-flux density in the plane *xy* in forward (a) and backward (b) flows. q, W/m^2; x, y, m.

are the farthest from the heat-transfer wall. The profile of the dependences T_i (i = 1, ..., 5) in any transverse cross section is qualitatively the same along the whole length of the heat exchanger and does not depend on the regime of heat-carrier flow (forward or backward flow). It is reasonable to assume that T_2 and T_4 in the discharge branch pipes are equal to their mean values, since the liquid which leaves the heat exchanger is no longer subject to the effect of the other liquid and its temperature over the cross section can be taken constant.

In calculations we used the parameters of the setup and experimental data of [14]. The results obtained are in satisfactory agreement with experimental data. This is explained partially by the fact that a smooth plate was used as a partition, whereas, as is known, plate surfaces in real plate heat exchangers are corrugated. Corrugations enlarge the surface area, thus facilitating heat-transfer enhancement.

By the obtained curves of temperature distribution in the cross section of the device we can explain the sharp jumps of liquid temperature at the outlet from the supply branch pipes on its entry to the heat exchanger, $[T_2^* - T_2]$ and $[T_4^* - T_4]$, which were obtained in [5] by using mixed boundary conditions [15]. Earlier [1–8, 14], section-mean temperature was used in calculation, which, as is seen from Fig. 2, is not equal to the temperature in the supply branch pipe. The value of the jumps $[T_2^* - T_2]$ and $[T_4^* - T_4]$ will depend on the point of the cross section where T_2 and T_4 were calculated.

It follows from the graphs of heat-flux densities (Figs. 3 and 4) that the backward flow is more constant along the length of the apparatus than the forward flow. It is also obtained that section-mean temperature is determined at about 2/3 of the channel depth rather than at the center of the channel; therefore, if it is necessary to measure this temperature a thermometer must be placed at this depth.

This model allows calculation of all heat fluxes in different directions and temperature at any point of a heat exchanger.

NOTATION

a_j, coefficients of thermal diffusivity, m^2/sec; c_j, specific heat capacities, J/(kg·K); E_{10}–E_0, coefficients of the characteristic equation; f, possible heat sources, W; h_i (i = 1, 3, 5) and h_j (j = 2, 4), thicknesses of the walls and layers of the heat carrier, m; H_k (k = 1, ..., 5), coordinates of the upper boundary of the corresponding layer; L, linear dimension characterizing the length of the heat-transfer zone along the apparatus, m; $m_i(x)$, $b_i(x)$, $g_i(x)$ (i = 1, ..., 5), coefficients of the Taylor series; $M_i(x)$, $B_i(x)$, $A_i(x)$, partial solution of a homogeneous system; M_{i0}, B_{i0}, A_{i0}, coefficients of partial solutions of a homogeneous system; T_i (i = 1, 3, 5) and T_j (j = 2, 4), temperature at the corresponding points of the walls and layers of the heat carrier, K; T_0, temperature of the surrounding air, K; T_2^* and T_4^*, temperatures of the hot and cold liquids in the supply branch pipes, respectively, K; v, projection of the velocity of liquid to the x axis in the jth channel, m/sec; q, heat-flux density in the transverse direction, W/m^2; x and y, current longitudinal and transverse coordinates, m; y_i^* (i = 1, ..., 5), coordinates of the center of the corresponding layer, m; α_i (i = 1, 3, 5), coefficients of heat transfer from walls to air, W/(m^2·K); λ_i (i = 1, ..., 5), coefficients of thermal conductivity in the corresponding layers, W/(m·K); ρ_j, densities of liquids, kg/m^3. Subscripts: i = 1, 3, 5, numbers of walls of the heat exchanger; j = 2, 4, numbers of layers of the heat carrier; (n), values of the coefficients of partial solutions of the homogeneous system with the corresponding characteristic roots; ′, first derivative; ″, second derivative.

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